

# On Duru-Kleinert Path Integral in Quantum Cosmology

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## Abstract

We show that the Duru-Kleinert fixed energy amplitude leads to the path integral for the propagation amplitude in the closed FRW quantum cosmology with scale factor as one degree of freedom. Then, using the Duru-Kleinert equivalence of corresponding actions, we calculate the tunneling rate, with exact prefactor, through the dilute-instanton approximation to first order in  $\hbar$ .

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# 1 Introduction

Tunneling is occurrent in almost all branches of physics including cosmology. Some of the most significant examples are, the decay of the QFT false vacuum in the inflationary model of universe [3], fluctuations as changes in the topology in semiclassical gravity via tunneling process [6], tunneling rates for the production of pairs of black holes [7], minisuperspace quantum cosmology [2], and etc.

Here in this article we are concerned with the calculation of the tunneling rate of the universe from nothing to the FRW universe based on the minisuperspace model with only one degree of freedom by applying the dilute-instanton approximation on the Duru-Kleinert path integral.

The Duru-Kleinert path integral formula for fixed energy amplitude is an alternative approach to handle systems with singular potentials [1]. The heart of this viewpoint is based on the Duru-Kleinert equivalence of actions, leading to the same fixed energy amplitude, by means of arbitrary regulating functions originating from the local reparametrization invariance. On the other hand, reparametrization invariance is a basic property of general relativity [2], so it is expected that:

One can establish a relation between Duru-Kleinert path integral for fixed zero energy amplitude and the standard path integral in quantum cosmology, and by using the corresponding Duru-Kleinert equivalence of actions it is possible to work with the action that contains the standard quadratic kinetic term instead of non-standard one.

In this paper we have studied these two subjects in the context of a “mini-superspace” model with only one degree of freedom. In section 2, the Duru-Kleinert path integral formula and Duru-Kleinert equivalence of corresponding actions is briefly reviewed. In section 3, the standard path integral in quantum cosmology and its relation to Duru-Kleinert path integral for closed FRW cosmology with only one degree of freedom (the scale factor) is investigated. This section ends by introducing an equivalent standard quadratic action for this cosmology. Finally in section 4, the rate of tunneling from nothing to the FRW universe is calculated through the dilute instanton approximation to first order in  $\hbar$  [3], where its prefactor is calculated by the heat kernel method [4], using the shape invariance symmetry [5].

## 2 Duru-Kleinert equivalence

In this section we briefly review Ref. 1. The fundamental object of path integration is the time displacement amplitude or propagator of a system,  $(X_b t_b | X_a t_a)$ . For a system with a time independent Hamiltonian, the object  $(X_b t_b | X_a t_a)$  supplied by a path integral is the causal propagator

$$(X_b t_b | X_a t_a) = \theta(t_a - t_b) \langle X_b | \exp(-i\hat{H}(t_b - t_a)/\hbar) | X_a \rangle. \quad (1)$$

Fourier transforming the causal propagator in the time variable, we obtain the fixed energy amplitude

$$(X_b | X_a)_E = \int_{t_a}^{\infty} dt_b e^{iE(t_b - t_a)/\hbar} (X_b t_b | X_a t_a) \quad (2)$$

This amplitude contains as much information on the system as the propagator  $(X_b \ t_b \mid X_a \ t_a)$ , and its path integral form is as follows:

$$(X_b \mid X_a)_E = \int_{t_a}^{\infty} dt_b \int \mathcal{D}x(t) e^{i\mathcal{A}_E/\hbar} \quad (3)$$

with the action

$$\mathcal{A}_E = \int_{t_a}^{t_b} dt \left[ \frac{M}{2} \dot{x}^2(t) - V(x(t)) + E \right] \quad (4)$$

where  $\dot{x}$  denotes the derivatives with respect to  $t$ . In Ref. 1, it has been shown that fixed energy amplitude (3) is equivalent with the following fixed energy amplitude,

$$(X_b \mid X_a)_E = \int_0^{\infty} dS [f_r(x_b) f_l(x_a) \int \mathcal{D}x(s) e^{i\mathcal{A}_E^f/\hbar}] \quad (5)$$

with the action

$$\mathcal{A}_E^f = \int_0^S ds \left\{ \frac{M}{2f(x(s))} x'^2(s) - f(x(s)) [V(x(s)) - E] \right\} \quad (6)$$

where  $f_r$  and  $f_l$  are arbitrary regulating functions and  $x'$  denotes the derivatives with respect to  $s$ .

The actions  $\mathcal{A}_E$  and  $\mathcal{A}_E^f$ , both of which lead to the same fixed-energy amplitude  $(X_b \mid X_a)_E$  are called Duru-Kleinert equivalent<sup>1</sup>. The motivation of Duru and Kleinert, using this equivalence, was to investigate the path integrals for singular potentials.

In the following section we show that one can use this equivalence to investigate the quantum cosmological models, with one degree of freedom. To see this rewrite the action  $\mathcal{A}_E^f$  in a suitable form such that it describes a system with zero energy; as only in this sense can we describe a quantum cosmological model with zero energy.

Imposing  $E = 0$  in (6), with a simple manipulation, gives

$$\mathcal{A}_E^f = \int_0^1 ds' S f(X(s')) \left\{ \frac{M}{2[Sf(X(s'))]^2} \dot{X}^2(s') - V(X(s')) \right\} \quad (7)$$

where  $\dot{X}$  denotes the derivative with respect to new parameter  $s'$  defined by

$$s' = S^{-1}s \quad (8)$$

with  $S$  as a dimensionless scale parameter.

After a Wick rotation  $s' = -i\tau$ , we get the required Euclidean action and the path integral

$$I_0^f = \int_0^1 d\tau S f(X(\tau)) \left\{ \frac{M}{2[Sf(X(\tau))]^2} \dot{X}^2(\tau) + V(X(\tau)) \right\} \quad (9)$$

$$(X_b \mid X_a) = \int_0^{\infty} dS [f_r(X_b) f_l(X_a) \int \mathcal{D}X(\tau) e^{-I_0^f/\hbar}]. \quad (10)$$

where  $\tau$  is the Euclidean time. We will use eqs. (9) and (10) in the following section.

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<sup>1</sup>Of course a third action  $\mathcal{A}_{E,\varepsilon}^{DK}$  is also Duru-Kleinert equivalent of  $\mathcal{A}_E$  and  $\mathcal{A}_E^f$  which we do not consider here.

### 3 Path integral in Quantum Cosmology

General formalism of quantum cosmology is based on the Hamiltonian formulation of general relativity, specially Dirac quantization procedure in which the wave function of the universe  $\Psi$  is obtained by solving the Wheeler-DeWitt equation

$$\hat{H}\Psi = 0. \quad (11)$$

A more general and more powerful tool for calculating the wave function is the path integral. In ref. 2 it is shown that the path integral for the propagation amplitude between fixed initial and final configurations can be written as

$$(X_b | X_a) = \int_0^\infty dN < X_b, N | X_a, 0 > = \int_0^\infty dN \int \mathcal{D}X e^{-I[X(\tau), N]/\hbar} \quad (12)$$

where  $< X_b, N | X_a, 0 >$  is a Green function for the Wheeler-DeWitt equation and  $N$  is the lapse function. The Euclidean action  $I$  is defined on minisuperspace in the gauge  $\dot{N} = 0$  as

$$I[X(\tau), N] = \int_0^1 d\tau N \left[ \frac{1}{2N^2} f_{ab}(X) \dot{X}^a \dot{X}^b + V(X) \right] \quad (13)$$

where  $f_{ab}(X)$  is the metric defined on minisuperspace and has indefinite signature.

Here we consider a model in which the metric  $f_{ab}(X)$  is defined by only one component and takes the following Euclidean action [2],

$$I = \int_0^1 d\tau N \left[ \frac{R\dot{R}^2}{2N^2} + \frac{1}{2} \left( R - \frac{R^3}{a_0^2} \right) \right] \quad (14)$$

where  $R$  is the scale factor and  $R_0^2 = \frac{3}{\Lambda}$  is interpreted as the minimum radius of the universe after tunneling from nothing [2] ( $\Lambda$  is cosmological constant). This model describes the closed FRW universe with one degree of freedom  $R$ .

Now we rewrite the action (14) as

$$I = \int_0^1 d\tau N R^{-1} \left[ \frac{\dot{R}^2}{2N^2 R^{-2}} + \frac{1}{2} \left( R^2 - \frac{R^4}{R_0^2} \right) \right]. \quad (15)$$

Comparing this action with (9) (with  $M = 1$ ) we find that by choosing

$$NR^{-1} = Sf(R) \quad (16)$$

we obtain (9) in the form

$$I = I_0^f = \int_0^1 d\tau Sf(R) \left[ \frac{\dot{R}^2}{2[Sf(R)]^2} + V(R) \right] \quad (17)$$

such that

$$V(R) = \frac{1}{2} \left( R^2 - \frac{R^4}{R_0^2} \right). \quad (18)$$

The gauge  $\dot{N} = 0$  gives

$$f(R) = CR^{-1} \quad (19)$$

where  $C$  is a constant which we set to  $C = R_a^{-1}$  so that

$$S = N R_a.$$

Now, one can show that the path integral (10) corresponds to the path integral (12).

To see this, assume

$$f_r(R) = 1 \quad , \quad f_l(R) = f(R) \quad (20)$$

so that the path integral (10) can be written as

$$(R_b | R_a) = \int_0^\infty dN \int \mathcal{D}R e^{-I_0^f/\hbar} \quad (21)$$

where  $I_0^f$  is given by (17). This shows that the Duru-Kleinert path integral (21) is exactly in the form of (12) as a path integral for this cosmological model. Now, using the Duru-Kleinert equivalence, we can work with the standard quadratic action

$$I_0 = \int_{\tau_a}^{\tau_b} d\tau \left[ \frac{1}{2} \dot{R}^2(\tau) + \frac{1}{2} \left( R^2 - \frac{R^4}{R_0^2} \right) \right] \quad (22)$$

instead of the action (17) or (14), where a Wick rotation with  $E = 0$  has also been used in the equation (4).

## 4 Tunneling rate

The Euclidean type Lagrangian corresponding to the action (22) has the following quadratic form

$$L_E = \frac{1}{2} \dot{R}^2 + \frac{1}{2} \left( R^2 - \frac{R^4}{R_0^2} \right). \quad (23)$$

The corresponding Hamiltonian is obtained by a Legendre transformation

$$H_E = \frac{\dot{R}^2}{2} - \frac{1}{2} \left( R^2 - \frac{R^4}{R_0^2} \right). \quad (24)$$

Imposing  $H_E = 0$  gives a nontrivial “instanton solution” as

$$R(\tau) = \frac{R_0}{\cosh(\tau)}, \quad (25)$$

which describes a particle rolling down from the top of a potential  $-V(R)$  at  $\tau \rightarrow -\infty$  and  $R = 0$ , bouncing back at  $\tau = 0$  and  $R = R_0$  and finally reaching the top of the potential at  $\tau \rightarrow +\infty$  and  $R = 0$ .

The region of the barrier  $0 < R < R_0$  is classically forbidden for the zero energy particle, but quantum mechanically it can tunnel through it with a tunneling probability which is calculated making use of the instanton solution (25).

The quantized FRW universe is mathematically equivalent to this particle, such that the particle at  $R = 0$  and  $R = R_0$  represents “nothing” and “FRW” universes respectively. Therefore one can find the probability

$$| \langle FRW(R_0) | nothing \rangle |^2.$$

The rate of tunneling  $\Gamma$  is calculated through the dilute instanton approximation to first order in  $\hbar$  as [3]

$$\Gamma = \left[ \frac{\det'(-\partial_\tau^2 + V''(R))}{\det(-\partial_\tau^2 + \omega^2)} \right]^{-1/2} e^{-\frac{I_0(R)}{\hbar}} \left[ \frac{I_0(R)}{2\pi\hbar} \right]^{1/2} \quad (26)$$

where  $\det'$  is the determinant without the zero eigenvalue,  $V(R)''$  is the second derivative of the potential at the instanton solution (25),  $\omega$  corresponds to the real part of the energy of the false vacuum ( $|nothing\rangle$ ) and  $I_0(R)$  is the corresponding Euclidean action. The determinant in the numerator is defined as

$$\det'[-\partial_\tau^2 + V''(R)] \equiv \prod_{n=1}^{\infty} |\lambda_n| \quad (27)$$

where  $\lambda_n$  are the non-zero eigenvalues of the operator  $-\partial_\tau^2 + V''(R)$ . The explicit form of this operator is obtained as

$$O \equiv \left[ -\frac{d^2}{dx^2} + 1 - \frac{6}{\cosh^2(x)} \right] \quad (28)$$

where we have used Eqs.(18) and (25) and a change of variable  $x = \tau$  has been done.

Now, we can calculate the ratio of the determinants as follows: First we explain very briefly how one can calculate the determinant of an operator through the heat kernel method [4]. We introduce the generalized Riemann zeta function of the operator  $A$  by

$$\zeta_A(s) = \sum_m \frac{1}{|\lambda_m|^s} \quad (29)$$

where  $\lambda_m$  are eigenvalues of the operator  $A$ , and the determinant of the operator  $A$  is given by

$$\det A = e^{-\zeta'_A(0)}. \quad (30)$$

On the other hand  $\zeta_A(s)$  is the Mellin transformation of the heat kernel  $G(x, y, t)$ <sup>2</sup> which satisfies the following heat diffusion equation,

$$A G(x, y, t) = -\frac{\partial G(x, y, t)}{\partial t} \quad (31)$$

with an initial condition  $G(x, y, 0) = \delta(x - y)$ . Note that  $G(x, y, t)$  can be written in terms of its spectrum

$$G(x, y, t) = \sum_m e^{-\lambda_m t} \psi_m^*(x) \psi_m(y). \quad (32)$$

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<sup>2</sup>Here  $t$  is a typical time parameter.

An integral is written for the sum if the spectrum is continuous. From relations (30) and (31) it is clear that

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \int_{-\infty}^{+\infty} dx G(x, x, t). \quad (33)$$

Now, in order to calculate the ratio of the determinants in (26), called a prefactor, we note that it is required to find the difference of the functions  $G(x, y, t)$ .

We rewrite the operator (28) as:

$$\left[ \left( -\frac{d^2}{dx^2} - \frac{2(2+1)}{\cosh^2(x)} + 4 \right) - 3 \right]. \quad (34)$$

This is the same as the operator which appears in Ref.5, for values of  $l = 2$ ,  $h = -3$ ; so the heat kernel  $G(x, y, t)$  corresponding to the operator (34) is given by

$$G_{\Delta_2(0)-3}(x, y, t) = \frac{e^{-(4-3)t}}{2\sqrt{\pi t}} e^{-(x-y)^2/4t} \quad (35)$$

and

$$\begin{aligned} G_{\Delta_2-3}(x, y, t) &= \psi_{2,0}^*(x) \psi_{2,0}(y) e^{-| -3 | t} \\ &+ \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{(1+k^2)t}}{(k^2+1)(k^2+4)} \left[ (B_2^\dagger(x) B_1^\dagger(x) e^{-ikx}) \right] \left[ (B_2^\dagger(y) B_1^\dagger(y) e^{iky}) \right]. \end{aligned} \quad (36)$$

The functions  $\psi_{l,m}$  and  $\psi_{l,k}$  are the eigenfunctions corresponding to discrete spectrum  $E_{l,m} = m(2l - m)$  and continuous spectrum  $E_{l,k} = l^2 + k^2$  of the following operator

$$-\frac{d^2}{dx^2} - \frac{l(l+1)}{\cosh^2(x)} + l^2$$

respectively, and are given by [5]

$$\psi_{l,m}(x) = \sqrt{\frac{2(2m-1)!}{\prod_{j=1}^m j(2l-j)}} \frac{1}{2^m(m-1)!} B_l^\dagger(x) B_{l-1}^\dagger(x) \cdots B_{m+1}^\dagger(x) \frac{1}{\cosh^m(x)}$$

and

$$\psi_{l,k}(x) = \frac{B_l^\dagger(x)}{\sqrt{k^2 + l^2}} \frac{B_{l-1}^\dagger(x)}{\sqrt{k^2 + (l-1)^2}} \cdots \frac{B_1^\dagger(x)}{\sqrt{k^2 + 1^2}} \frac{e^{ikx}}{\sqrt{2\pi}}$$

where

$$B_l(x) := \frac{d}{dx} + l \tanh(x), \quad B_l^\dagger(x) := -\frac{d}{dx} + l \tanh(x).$$

Now, we can write

$$\int_{-\infty}^{+\infty} dx [G_{\Delta_2-3}(x, x, t) - G_{\Delta_2(0)-3}(x, x, t)] = e^{-3t} - \frac{3}{\pi} e^{-t} \int_{-\infty}^{+\infty} dk \frac{(k^2 + 2)e^{-k^2 t}}{(k^2 + 1)(k^2 + 4)}$$

and

$$\begin{aligned}
\zeta_{\Delta_2-3}(s) - \zeta_{\Delta_2(0)-3}(s) &= \frac{1}{3^s} - \frac{3}{\pi} \int_{-\infty}^{+\infty} \frac{dk}{(k^2+1)^s(k^2+4)} - \frac{3}{\pi} \int_{-\infty}^{+\infty} \frac{dk}{(k^2+1)^{s+1}(k^2+4)} \\
&= \frac{1}{3^s} - \frac{3}{\sqrt{\pi}} \frac{1}{2^{2s+1}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \left\{ F\left(s, s+\frac{1}{2}, s+1; \frac{3}{4}\right) \right. \\
&\quad \left. + \frac{2s+1}{8(s+1)} F\left(s+1, s+\frac{3}{2}, s+2; \frac{3}{4}\right) \right\}.
\end{aligned}$$

Thus, we have

$$\zeta_{\Delta_2-3}(0) - \zeta_{\Delta_2(0)-3}(0) = -1, \quad \zeta'_{\Delta_2-3}(0) - \zeta'_{\Delta_2(0)-3}(0) = Ln(12)$$

So we get the following value for the prefactor in (26)

$$prefactor = e^{-ln(12)} = \frac{1}{12} \quad (37)$$

Substituting the result (37) in Eq. (26) gives the tunneling rate of the universe from nothing to the FRW universe

$$\Gamma = \frac{2}{\sqrt{\pi\hbar}} R_0 e^{-\frac{2R_0^2}{3\hbar}} + O(\hbar).$$

in a good agreement with the result obtained by WKB approximation (Atkatz [2]).

## 5 Conclusions

We have shown in this paper that one can obtain the path integral formula of quantum cosmology by Duru-Kleinert path integral formula, at least for a model with one degree of freedom. The prime point is that, to the extent the path integral quantum cosmology is concerned, one can work with the standard action instead of non standard one, by using the Duru-Kleinert equivalence of the actions. This will be valuable in simplifying the possible technical problems which may appear in working with non standard actions. We have concentrated on the model with only one degree of freedom. Whether this procedure works for higher degrees of freedom is a question which requires further investigation.

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